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Time to Reach Stationarity in the Bernoulli–Laplace Diffusion Model with Many Urns

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1. INTRODUCTION

Consider m urns labelled 1 thorugh m and nm balls labelled 1 through nm. Put the balls labelled 1 through n in the first urn the balls labelled $n+1$ through $2n$ in the second urn, and so on. Then at each time choose at random two balls that belong to different urns and switch them. This model for $m = 2$ was introduced by Bernoulli and Laplace [9]. Diaconis and Shahshahani [8] studied this case using the spherical Fourier transform of the Gelfand pair $(S_{2n}, S_n \times S_n)$, where S_n is the symmetric group. They gave an upper bound and a lower bound for the variation distance of the distribution probability after k steps from the stationary distribution showing that for such a model $k = \frac{1}{4}n \log n$ switches are necessary and sufficient to reach stationarity, i.e., to mix the urns. More precisely, their bounds show that for fewer than $\frac{1}{4}n \log n$ switches such variation distance is near its maximum 1; after $\frac{1}{4}n \log n$ switches it tends to zero exponentially fast. This is an example of the so-called cutoff phenomenon (see also [3, pp. 23–24]). They also studied the case $n = 1$ (with a slightly different measure in order to avoid parity problems) using the character theory of the symmetric group; they showed that $\frac{1}{2}m \log m$ random transpositions generate a random permutation [7]. In [3, p. 59], Diaconis posed the problem of computing the time to reach stationarity in the case $m = 3$. In the present paper this problem is solved for every $n, m \geq 2$ showing that for such values of n and m the model reaches stationarity after $\frac{1}{4}n(m-1)\log(nm^2)$ switches. We give lower and upper bounds that show a cutoff phenomenon as in [7] and [8]. We give an upper bound for m fixed and another for n fixed. The methods employed are similar to those developed by Diaconis and Shahshahani [7, 8], but there is something new. In [7] the

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measure that describes the process is in the centre of the group algebra of the symmetric group and in [8] the measure belongs to the commutative algebra of bi-K-invariant functions of the Gelfand pair $(S_{2n}, S_n \times S_n)$. In the present paper the measure that describes the process belongs to the centre of a noncommutative algebra of bi-invariant functions.

Another important generalization of the Bernoulli–Laplace diffusion model is in Greenhalgh's thesis [11]. He studied a model made up of n balls labelled $1, 2, \ldots, n$, $n - k$ balls in a rack in order $1, 2, \ldots, n - k$, and the others in a bag. At each time one ball is picked from the bag and one from the rack and they are switched. Greenhalgh showed that $n \log n$ steps mix things up. His measure belongs to the algebra of functions on S_n that are conjugacy invariant under $S_{n-k} \times S_k$ and bi-invariant under S_k. Such algebra is commutative and this leads to a suitable spherical Fourier transform that diagonalizes the measure.

The plan of this paper is as follows. In Section 2 we compute the Fourier transform of the measure that describes the process, using a trick that we learned in [11]. In Section 3 we give two "upper bounds for the rate of convergence to the stationary distribution, along the lines of [7], [8], and especially [3, pp. 36–43]. In Section 4 we compute some spherical characters and spherical matrix coefficients for the symmetric group, which are used in Section 5 to give a lower bound along the lines of [8].

In [3] there is a good exposition of the theory of Fourier transform on finite groups with applications to probability problems, including the results obtained in [7] and [8]. Another nice survey is in [4]. In [10], [12], and [14] one can find more complete expositions of the representation theory of the symmetric group. Unexplained terminology is as in these books.

2. A FOURIER TRANSFORM

In this section we compute the Fourier transform of the measure that describes the process. We use a simple trick taken from [11], where it is applied to a different measure. More generally, consider a model with m urns: the first urn contains a_1 balls labelled 1 through a_1, the second urn contains a_2 balls labelled $a_1 + 1$ through $a_1 + a_2$, and so on. Let $N = a_1 + \cdots + a_m$. In such a model at each time two balls that belong to different urns are chosen at random and switched. The probability of choosing one ball in urn i and the other in urn j is exactly $a_i a_j / \sum_{h<k} a_h a_k$. Think of the symmetric group S_N as the group of all permutations of the balls in the urns. Let S_{a_i} be the symmetric group of all permutations of the balls labelled $a_1 + \cdots + a_{i-1} + 1$ through $a_1 + \cdots + a_i$, for $i = 1, \ldots, m$, that is, S_{a_i} is the group of permutations of the balls initially in the ith urn.

Then $K = S_{a_1} \times \cdots \times S_{a_m}$ is the stabilizer of the initial configuration of the process. Every configuration of the model is given by a sequence $(A_1, \dots, A_m)$ of subsets of $\{1, \dots, N\}$ such that for $i = 1, \dots, m$ the cardinality of A_i (that represent the content of the urn i) is a_i and, for $i \neq j$, $A_i \cap A_j = \emptyset$ (so $A_1 \cup \cdots \cup A_m = \{1, \dots, N\}$). The space of all configurations may be identified with the homogeneous space S_N/K of left cosets mod K. Let T be the set of all transpositions in S_N, $T_1 = T \cap K$ and $T_2 = T \setminus T_1$. So if $t \in T_1$, then t switches two balls that are in the same urn at the beginning of the process; if $t \in T_2$, the balls switched by t are in different urns at the beginning. Observe that there is only one way to write an element of $T_2 K$ as a product tk with $t \in T_2$ and $k \in K$. So we can define a probability measure μ on S_N by $\mu(g) = 1/(|T_2|\,|K|)$ if $g \in \{tk\colon\ t \in T_2,\ k \in K\}$ and $\mu(g) = 0$ otherwise. Because $kT_2k^{-1} = T_2$ for $k \in K$, μ is bi-K-invariant. Moreover, the process may be seen as the random walk on S_N/K induced by μ, i.e., if $x \in S_N/K$ and $x = gK$ with $g \in S_N$, the probability $P_k(x)$ that the model is in the configuration x after k steps is exactly $\mu^{*k}(gK)$, where μ^{*k} denotes the convolution $\mu * \cdots * \mu$ k times (compare with [3, pp. 51–54]).

Now let (ρ, V) be an irreducible representation of S_N and let h be the dimension of the space of K-invariant vectors in V. Choose an orthonormal basis in V such that the first h vectors are K-invariant (and so form a basis for the space of K-invariant vectors in V). Then in such a basis the operator $P_\rho = (1/|K|)\sum_{k \in K} \rho(k)$ is represented by the matrix $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where I is the $h \times h$ identity matrix. Let χ_ρ and d_ρ, respectively, denote the character and the dimension of ρ. Whereas T is a conjugacy class in S_N and χ_ρ is a class function, we may define $r(\rho)$ as the value of χ_ρ/d_ρ at any $t \in T$. Then it is not hard to prove that $(1/|T|)\sum_{t \in T} \rho(t) = r(\rho) I'$, where I' is the $d_\rho \times d_\rho$ identity matrix (see [3, p. 36] or [7]. See also the nice exposition in [4, pp. 43–48]). Using these facts, we can easily compute the Fourier transform of μ at the representation ρ:

$$\begin{aligned}
\hat{\mu}(\rho) &= \frac{1}{|T_2|\,|K|} \sum_{t \in T_2} \sum_{k \in K} \rho(tk) \\
&= \frac{1}{|T_2|\,|K|} \left[\sum_{t \in T} \sum_{k \in K} \rho(tk) - \sum_{t \in T_1} \sum_{k \in K} \rho(tk) \right] \\
&= \frac{1}{|T_2|} \left[\sum_{t \in T} \rho(t) P_\rho - |T_1| P_\rho \right] \\
&= \frac{1}{|T_2|} \left[|T| r(\rho) - |T_1| \right] P_\rho .
\end{aligned}$$

This shows that μ is in the centre of the algebra of the bi-K-invariant functions on S_N. The value of $r(\rho)$ is well known ([3, p. 40] or [10, p. 52]): if $\lambda = (\lambda_1, \dots, \lambda_h)$ is the partition of N canonically associated to ρ, then

$$r(\rho) = \frac{1}{N(N-1)} \sum_{j=1}^{h} \left[\lambda_j^2 - (2j-1)\lambda_j \right].$$

Moreover, $|T| = (N(N-1))/2$, $|T_1| = \sum_{i=1}^{m} (a_i(a_i - 1))/2$, and $|T_2| = \sum_{1 \le i \le j \le m} a_i a_j$. Thus for the Fourier transform of μ we obtain the expression

$$\hat{\mu}(\rho) = \left\{ 1 - \frac{N^2 - \sum_{j=1}^{h} \lambda_j \left[\lambda_j - 2(j-1) \right]}{N^2 - \sum_{i=1}^{m} a_i^2} \right\} P_\rho \tag{2.1}$$

for every irreducible representation ρ of S_N. In what follows, the quantity in curly brackets in (2.1) will be denoted by $q(\lambda)$.

Remark 1. There are other natural mixing processes for the Bernoulli–Laplace diffusion model with more than two urns. One of the most interesting is the following: At each time, choose a pair of urns at random, select a ball in each at random, and switch the balls. Then, except for the case $a_1 = a_2 = \cdots = a_m$, we obtain a different diffusion model. More generally, if $p(i, j)$, for $1 \le i < j \le m$, are nonnegative real numbers such that $\sum_{1 \le i < j \le m} p(i, j) = 1$, we can consider the diffusion model defined by the following mixing procedure: Choose at random two urns with the condition that the probability of choosing urn i and j is $p(i, j)$; then switch a random ball between each of the chosen urns. We can suppose that for every pair of urns i, j there exists a "path" $i_0 = i, i_1, \dots, i_w = j$ such that $p(i_s, i_{s+1}) > 0$ for $s = 0, \dots, w-1$, so that the process "mixes" all the model. If $T_{i,j}$ is the set of all transpositions in S_N that move one ball in urn i and the other in urn j, and K is as before, then this process is described by the probability measure ν on S_N defined by $\nu(g) = (p(i, j))/(a_i a_j |K|)$ if $g \in \{tK\colon k \in K \text{ and } t \in T_{i,j}\}$, for $1 \le i < j \le m$, and $\nu(g) = 0$ for all the others, $g \in S_N$. Clearly, ν is bi-K-invariant and symmetric [i.e., $\upsilon(g^{-1}) = \overline{\nu(g)}$]; it follows that the Fourier transform of ν is a self-adjoint operator and so is diagonalizable (in a suitable basis for the bi-K-invariant functions). However, such diagonalization requires a deeper development of the harmonic analysis on the algebra of bi-K-invariant functions on S_N. The case $m = 3$ is considered in [16].

Remark 2. Let B_N be the group of all isometries of an N-dimensional cube. Then [15] (S_{2N}, B_N) is a Gelfand pair and the homogeneous space S_{2N}/B_N may be identified with the set of all partitions of $2N$ elements into N unordered pairs. Diaconis [2] studied a natural random walk on

S_{2N}/B_N: at each step two pairs are picked at random and a random element is switched between them. He shows that $\frac{1}{2}N \log N$ switches are necessary and sufficient to mix the urns. The Fourier transform of the corresponding measure can be evaluated by the same trick we have used in this section.

3. UPPER BOUNDS CALCULATIONS

In this section, we restrict our attention to the case $a_1 = \cdots = a_m = n$ (so $N = nm$). We assume $n, m \geq 2$. As in [3], for P and Q probability measures on the space S_N/K we define the variation distance by $\|P - Q\| = \sup|P(A) - Q(A)|$, where the supremum is over all subsets A of S_N/K. If U is the uniform probability on S_N/K, i.e., $U(x) = |K|/N!$ for every $x \in S_N/K$, then the upper bound lemma in [3, p. 53] ensures that

$$\|P_k - U\|^2 \leq \tfrac{1}{4} \sum d_\rho \operatorname{Tr}\left(\hat{\mu}(\rho)^{2k} \right),$$

where the sum is over all nontrivial irreducible representations of S_N that occur in the decomposition of the permutation representation of S_N on S_N/K and d_ρ is the dimension of the representation ρ. For λ a partition of N, define m_λ as the multiplicity of the irreducible representation canonically associated to λ in such a decomposition (the coefficients m_λ are given by the so-called Young's rule that will be described and used in the sequel). Then from (2.1) and the above-mentioned upper bound lemma, it follows that

$$\|P_k - U\|^2 \leq \tfrac{1}{4} \sum m_\lambda d_\lambda q(\lambda)^{2k}, \tag{3.1}$$

where $q(\lambda)$ is as at the end of the last section and the sum is over all partitions $\lambda = (\lambda_1, \ldots, \lambda_h)$ of N such that $\sum_{i=1}^{j} \lambda_i \geq jn$, for $j = 1, \ldots, h$, because for other partitions $m_\lambda = 0$ [3, p. 134].

Consider now the term corresponding to $\lambda = (N-1, 1)$: $(N-1)(m-1)[1 - 2/(n(m-1))]^{2k}$. (In this case, $d_\lambda = N - 1$ [3, p. 136] and $m_\lambda = m - 1$ by Young's rule [3, pp. 138–139]). Using the inequality $\log(1 - x) \leq -x$, it is easy to prove that, for $k = \frac{1}{4}n(m-1)(\log(nm^2) + c)$ with $c > 0$, this term is bounded by e^{-c} and so it decreases exponentially. The rest of this section is devoted to proving that this is the slowest term in (3.1) and all other terms are geometrically smaller than it, and so after $k = \frac{1}{4}n(m-1)\log(nm^2)$ switches, $\|P_k - U\|$ decreases exponentially fast. Now we recall the definition of the majorization order in the set of all partitions of a fixed integer N [3, p. 131]: if $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\vartheta = (\vartheta_1, \ldots, \vartheta_s)$ are partitions of N, $\lambda \geq \vartheta$ means that $\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \vartheta_i$ for $j = 1, 2, \ldots$. The eigenvalues $q(\lambda)$ of $\hat{\mu}$ are monotone with respect to this order: if $\lambda \geq \vartheta$, then $q(\lambda) \geq q(\vartheta)$. The proof of this fact is like that of Lemma 1 in [3, p. 40]. This shows that the largest eigenvalue is $q((N-1,1)) = 1 - 2/(n(m-1))$ and the smallest is $q((n^m)) = -1/n$ [where (n^m) denotes the partition of nm in m parts of size n]. In the proofs of all the lemmas of this section, we also need the following simple inequality satisfied by the dimensions of the irreducible representations of the symmetric group: if $\lambda = (\lambda_1, \ldots, \lambda_h)$ is a partition of N, $\lambda_1 = N - j$, and $\lambda' = (\lambda_2, \ldots, \lambda_h)$, then [3, p. 40]

$$d_\lambda \leq \binom{N}{j} d_\lambda, \tag{3.2}$$

We recall that the Young (or Ferrers) diagram of a partition $\lambda = (\lambda_1, \ldots, \lambda_h)$ is the diagram that contains λ_1 squares in the first row, λ_2 squares in the second row, etc. It is also called the Young diagram of shape λ. The notation $\lambda \vdash N$ means that λ is a partition of N.

LEMMA 1. *Let $k = \frac{1}{4}n(m-1)(\log(nm^2) + c)$ with $c > 0$. Then there exists a universal constant a_1 such that*

$$\sum_{\lambda:\ q(\lambda) \leq 0} m_\lambda d_\lambda q(\lambda)^{2k} \leq a_1 e^{-c}.$$

Proof. Whereas $\max\{q(\lambda)^{2k}: q(\lambda) \leq 0\} = (1/n)^{2k}$ and $\sum_{\lambda \vdash N} m_\lambda d_\lambda = |S_N/K| = (nm)!/(n!)^m$, the sum in the statement is bounded above by $(nm)!/(n^{2k}(n!)^m)$. By Stirling's formula, if $k = \frac{1}{4}n(m-1)(\log(nm^2) + c)$ and $c > 0$, the last quantity is asymptotically equal to (for $nn = N \to \infty$)

$$\frac{m^{nm}(2\pi mn)^{1/2} \exp[-(1/2)n(m-1)c \log n]}{(2\pi n)^{m/2} \exp[(1/2)n(m-1)\log n \log(nm^2)]}$$

and this is smaller than e^{-c} if $n \geq 3$ and N is large. This proves the lemma for $n \geq 3$. Suppose now $n = 2$. Let ϑ be the partition $(3^{2m/3})$; then $q(\vartheta) = (1 - m/3)/(m-1)$. We split the sum into two parts, according $\lambda \geq \vartheta$ or not. Whereas $|q(\vartheta)| \leq \frac{1}{3}$, as in the case $n \geq 3$ it can be proved that

$$\sum_{\substack{q(\lambda) \leq 0 \\ \lambda \geq \vartheta}} m_\lambda d_\lambda q(\lambda)^{2k} \leq \left(\frac{1}{3}\right)^{2k} \frac{(2m)!}{2^m}.$$

For $k = (m-1)(\log m + c)$ the right-hand side of this inequality is asymptotic to

$$\frac{2^m m^{2m} (4\pi m)^{1/2}}{e^{2m} m^{2(m-1)\log 3}} \exp[-2(m-1)c \log 3]$$

and this is smaller than $\exp(-c)$ for m large. This bound the sum over all partitions with no more than $\frac{2}{3}m$ squares in the first column of the Young diagram. Now let P_h be the set of all partitions λ of N such that the first column of the associated Young diagram has h boxes and $q(\lambda) \leq 0$. If $m \geq h \geq \frac{2}{3}m$, then the smallest element in P_h with respect to the majorization order is the partition $\vartheta = (3^{2m-2h}, 2^{3h-2m})$, to which corresponds the eigenvalue

$$q(\vartheta) = 1 - \frac{4m^2 - 4mh - 6m + 3h + 3h^2}{2m(m-1)},$$

and $|q(\lambda)| \leq |q(\vartheta)|$ if $\lambda \in P_h$. Moreover, because $m_\lambda \leq d_\lambda$, using (3.2) with the first row replaced by the first column we can obtain

$$\sum_{\lambda \in P_h} d_\lambda m_\lambda \leq \binom{2m}{h}^2 \sum_{\lambda' \vdash 2m-h} d_{\lambda'}^2 = \binom{2m}{h}^2 (2m-h)! \leq \frac{(2m)^{4m-2h}}{(2m-h)!}.$$

From these facts it follows that if $m \geq 4$ and $k = (m-1)(\log m + c)$ with $c > 0$, then

$$\begin{aligned}
&\sum_{h=2m/3}^{m} \sum_{\lambda \in P_h} q(\lambda)^{2k} d_\lambda m_\lambda \\
&\quad \leq \sum_{h=2m/3}^{m} \frac{2^{4m-2h}}{(2m-h)!} m^{4m-2h} \exp(-c) \\
&\qquad \times \exp\left[\left(-4m + 6 + 4h - \frac{3h}{m} - \frac{3h^2}{m}\right) \log m\right] \\
&\quad \leq e^{-c} \sum_{h=2m/3}^{m} \frac{2^{4m-2h}}{(2m-h)!} m^4
\end{aligned}$$

and the last sum tends to zero if m tends to ∞. This completes the proof of the lemma. ∎

Now we recall Young's rule for the multiplicities m_λ (see [3, p. 138] or [12, 14] for proofs). If λ is a partition of N, the multiplicity m_λ equals the number of semistandard tableaux of shape λ and type (n^m), i.e., the number of placements of integers $\leq m$ into the Young tableau of shape λ, with numbers nondecreasing in rows and strictly increasing down the columns, such that the number i occurs n times. In particular, for $n = 1$ we obtain a combinatorial characterization of the dimension d_λ: if λ is a partition of N, d_λ is equal to the number of standard tableaux of shape λ, i.e., the number of placements of the integers $\{1, \dots, N\}$ into the Young diagram of λ with numbers strictly increasing both in rows and down the columns (see [12] or [14]). This fact was used by Diaconis and Shahshahani to prove (3.1). If we define Q_j as the set of all partitions λ of N such that the greatest part λ_1 is equal to $N - j$ and $q(\lambda)$ is nonnegative, then it is not hard to obtain the following bound from the Young's rule:

$$\text{if } \lambda \in Q_j, \text{ then } m_\lambda \le m^j. \tag{3.3}$$

This will be used in the proof of the following lemma.

LEMMA 2. *For every $t \in (0, \frac{1}{8})$ there exists a positive constant a_2 (not depending on n or m) such that if $k = \frac{1}{4} n(m-1)(\log(nm^2) + c)$ with $c > 0$, then*

$$\sum_{j=1}^{tmn} \sum_{\lambda \in Q_j} d_\lambda m_\lambda q(\lambda)^{2k} \le a_2 e^{-c}.$$

Proof. We proceed as in the first part of the proof of Theorem 5 of [3, pp. 41–42]. If $j < N/2$, the greatest partition in Q_j is $(N-j, j)$. Because $q(\lambda)$ is a monotone function, it follows that

$$q(\lambda) \le q((N-j, j)) = 1 - \frac{2j(N-j+1)}{n^2 m(m-1)}$$

when $\lambda \in Q_j$ and $j < N/2$. Moreover, from (3.2) and (3.3) we deduce that

$$\sum_{\lambda \in Q_j} d_\lambda m_\lambda \le m^j \binom{N}{j} \sum_{\lambda' \vdash j} d_{\lambda'},$$

where the last sum is over all the partitions λ' of j. However, such sum equals the number of involutions in S_j and is asymptotically equal to $(j/e)^{j/2}(\exp(j^{1/2}))/(2^{1/2}e^{1/4})$ (see [17, p. 267] and [1, 13] for proofs). From these facts it follows that there exists a positive constant A such that for every $n, m \ge 2$,

$$\sum_{j=1}^{tmn} \sum_{\lambda \in Q_j} q(\lambda)^{2k} m_\lambda d_\lambda \le A \sum_{j=1}^{tmn} \frac{m^j (nm)!}{(mn-j)!j!} \times \left(\frac{j}{e}\right)^{j/2} \exp\left(j^{1/2}\right) \left[1 - \frac{2j(nm-j+1)}{n^2 m(m-1)}\right]^{2k}.$$

However, if $k = \frac{1}{4}n(m-1)[\log(nm^2) + c]$ and $c > 0$, then

$$\left[1 - \frac{2j(nm - j + 1)}{n^2 m(m-1)}\right]^{2k} \le (nm^2)^{-1} e^{-c} (nm^2)^{-(nm-j)(j-1)/nm}.$$

Hence to prove the lemma it is sufficient to bound the sum

$$\sum_{j=1}^{tmn} \frac{(nm-1)!m^{j-1}}{(nm-j)!j!} \left(\frac{j}{e}\right)^{j/2} \exp(j^{1/2})(nm^2)^{-(nm-j)(j-1)/nm},$$

which is smaller than

$$\sum_{j=1}^{tmn} \frac{\exp(j^{1/2})}{j!} \left(\frac{j}{e}\right)^{j/2} (nm^2)^{j(j-1)/nm}.$$

In the last sum the ratio between two consecutive terms is

$$\exp\left[\frac{1}{(j)^{1/2} + (j+1)^{1/2}}\right]\left[\left(1 + \frac{1}{j}\right)^j \frac{1}{e(j+1)} (nm^2)^{4j/nm}\right]^{1/2}.$$

For $j = 1$ and nm large this ratio is smaller than $q < 1$. For $j \ge 2$ it is not greater than $\exp(1/(\sqrt{2} + \sqrt{3}))[1/(j+1)(nm^2)^{4j/nm}]^{1/2}$ and this, as a function of j, is decreasing if $j < nm/(4\log(nm^2))$ and increasing if $j > nm/(4\log(nm^2))$. So if for both $j = 2$ and $j = tnm$ the foregoing function is smaller than $q < 1$, we may bound the sum by $1/(1-q)$. It is true if $j = 2$ and nm is large. For $j = tnm$ such quantity is not greater than $2(t)^{-1/2} n^{2t-1/2} m^{4t-1/2}$, which is smaller than 1 if nm is large, because $t < 1/8$. This completes the proof of the lemma. ∎

The following two lemmas give bounds for the remaining part of the sum in (3.1); the first lemma is for m fixed and the second for n fixed. In the proofs of both lemmas we use the following inequality:

$$\text{if } \lambda \in Q_j, \text{ then } q(\lambda) \le 1 - \frac{j}{n(m-1)}. \tag{3.4}$$

The proof is straightforward: if $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition of N and $\lambda_1 = N - j$, then

$$q(\lambda) = 1 - \frac{N^2 - \sum_{i \ge 1}\lambda_i^2 + 2\sum_{i \ge 1}\lambda_i(i-1)}{n^2 m(m-1)} \le 1 - \frac{N^2 - (N-j)N}{n^2 m(m-1)}$$

$$= 1 - \frac{j}{n(m-1)}.$$

LEMMA 3. *Fix $t \in (0,1)$ and $m \geq 2$. Then there exists a positive constant a_3 not depending on n such that if $k = \frac{1}{4}n(m-1)(\log(nm^2)+c)$ with $c > 0$, then*

$$\sum_{j=tnm}^{n(m-1)} \sum_{\lambda \in Q_j} d_\lambda m_\lambda q(\lambda)^{2k} \leq a_3 e^{-c}.$$

Proof. We proceed as in the second part of the proof of Theorem 5 in [3, pp. 42–43]. Moreover, following the same path of the preceding lemma, the proof may be reduced to bounding

$$\sum_{j=tnm}^{n(m-1)} \frac{(nm)!}{(nm-j)!j!} m^j \left(\frac{j}{e}\right)^{j/2} \exp(j^{1/2}) \left(1 - \frac{j}{n(m-1)}\right)^{2k}$$

for $k = \frac{1}{4}n(m-1)(\log(nm^2)+c)$, $c > 0$, and m fixed. However, if $j \geq tnm$ and k has this value, then

$$\left(1 - \frac{j}{n(m-1)}\right)^{2k} \leq (1-t)^{cn(m-1)/2} \left(1 - \frac{j}{n(m-1)}\right)^{[n(m-1)\log(nm^2)]/2}$$

and $(1-t)^{cn(m-1)}$ is smaller than e^{-c} if n is large enough. So all we have to do is to bound the sum

$$\sum_{j=tnm}^{n(m-1)} \frac{(nm)!m^j}{(nm-j)!j!} \left(\frac{j}{e}\right)^{j/2} \exp(j^{1/2}) \left(1 - \frac{j}{n(m-1)}\right)^{[n(m-1)\log(nm^2)]/2}.$$

The ratio between two consecutive terms is not greater than

$$\exp\left[\frac{1}{(j)^{1/2}+(j+1)^{1/2}}\right] \frac{(nm-j)m}{(j+1)^{1/2}} \left[1 - \frac{1}{n(m-1)-j}\right]^{[n(m-1)\log(nm^2)]/2},$$

which is decreasing as a function of j. The first of these ratios (i.e., for $j = tmn$) is dominated by $n^{1/2}n^{-1/[2(1-t)]} = n^{-t/[2(1-t)]}$ and so is less than 1 for n large. So we may bound the sum by nm times the first term, that is, by

$$mn \frac{(nm)!}{[nm(1-t)]!(nmt)!} m^{nmt} \left(\frac{nmt}{e}\right)^{nmt/2} \exp\left[(tnm)^{1/2}\right]$$
$$\times \left(1 - \frac{tm}{m-1}\right)^{[n(m-1)\log(nm^2)]/2}.$$

Using Stirling's formula, it is not hard to see that this tends to zero if n tends to infinity, and this completes the proof of the lemma. ∎

In the proof of the next lemma we use the following bound for the multiplicities m_λ [which is different from (3.3)]: if $\lambda = (\lambda_1, \dots, \lambda_h)$ is a partition of nm and $\lambda_1 = nm - j$, then

$$m_\lambda \le \binom{m+j}{j} d_{\lambda'}, \tag{3.5}$$

where $\lambda' = (\lambda_2, \dots, \lambda_h)$. The proof of this fact is simple: To form a semistandard tableau of shape λ and type (n^m) we have to choose j numbers in the set $\{1, \dots, m\}$ and every number may be repeated at most n times. The number of possible choices is smaller than $\binom{m+j-1}{j}$, which is the number of j combinations of an m set with repetitions and this is smaller than $\binom{m+j}{j}$. For each choice of the j numbers, we have to form a semistandard tableau of shape λ'. The number of such tableaux is smaller than $d_{\lambda'}$, which is the number of standard tableaux of shape λ'. (Of course, not all of such semistandard tableaux of shape λ' combine with the first row to give a semistandard tableau of shape λ.) This completes the proof of the inequality. Now we can present the last lemma of this section.

LEMMA 4. *Fix $t \in (0,1)$ and $n \ge 2$. Then there exists a positive constant a_4 not depending on m such that if $k = \frac{1}{4}n(m-1)(\log(nm^2) + c)$ with $c > 0$, then*

$$\sum_{j=tnm}^{n(m-1)} \sum_{\lambda \in Q_j} d_\lambda m_\lambda q(\lambda)^{2k} \le a_4 e^{-c}.$$

Proof. Using (3.2) and (3.5) we obtain

$$\sum_{\lambda \in Q_j} d_\lambda m_\lambda \le \binom{m+j}{j}\binom{nm}{j} \sum_{\lambda' \vdash j} d_{\lambda'}^2 = \binom{m+j}{j}\binom{nm}{j} j!.$$

So

$$\sum_{j=tnm}^{n(m-1)} \sum_{\lambda \in Q_j} d_\lambda m_\lambda q(\lambda)^{2k} \le \sum_{j=tnm}^{n(m-1)} \binom{nm}{j}\binom{m+j}{j} j! \left[1 - \frac{j!}{n(m-1)}\right]^{2k}.$$

The rest of the proof consists in bounding the last written sum, and this may be done as in the previous lemma. ∎

Collecting together the bounds in Lemmas 1, 2, and 3 on the right side of (3.1) we obtain the upper bound for m fixed:

THEOREM 1. *Fix $m \geq 2$. Then there exists a positive constant b_1 not depending on n such that if $k = \frac{1}{4}n(m-1)(\log(nm^2)+c)$ and $c > 0$, then*

$$\|P_k - U\| \leq b_1 e^{-c/2}.$$

The upper bound for n fixed follows from Lemmas 1, 2, and 4:

THEOREM 2. *Fix $n \geq 2$. Then there exists a positive constant b_2 not depending on m such that if $k = \frac{1}{4}n(m-1)(\log(nm^2)+c)$ and $c > 0$, then*

$$\|P_k - U\| \leq b_2 e^{-c/2}.$$

We do not know if there exists an upper bound with a constant that is independent of both m and n. The proof of the existence of such a constant would require more refined bounds for the multiplicities m_λ.

4. SOME SPHERICAL CHARACTERS

In this section we use elementary representation theory of the symmetric group to derive some formulas that will be used in the next section to obtain a lower bound for our process. For every finite set X the scalar product of two complex valued functions f_1, f_2 defined on X will be defined by $\langle f_1, f_2\rangle = \sum_{x\in X} f_1(x)\overline{f_2(x)}$. Now we give a definition that generalizes the concepts of spherical function and of character.

DEFINITION 1. Let G be a finite group, K a subgroup of G, (ρ, V) a representation of G, and η the character of ρ. Then the K-spherical character η_K of ρ is defined by

$$\eta_K(g) = \frac{1}{|K|}\sum_{k\in K}\eta(gk)$$

for every g in G.

It is not hard to see that the spherical characters of the irreducible representations of G form an orthogonal basis for the centre of the algebra of bi-K-invariant functions on G. The proof of the following proposition is straightforward:

PROPOSITION 1. *Let $\xi_1, \dots, \xi_h$ be an orthonormal basis for the space of K-invariant vectors in V. Then the spherical character is given by*

$$\eta_K(g) = \sum_{i=1}^{h}\langle \rho(g)\xi_i, \xi_i\rangle.$$

In the next proposition we compute the spherical character of a permutation representation.

PROPOSITION 2. *Let G be a finite group acting transitively on a finite set X and K a subgroup of G. If $U_1, \ldots, U_h$ are the orbits of K on X, then the K-spherical character η of the permutation representation of G on X is given by*

$$\eta(g) = \sum_{i=1}^{h} \frac{1}{|U_i|} |gU_i \cap U_i|.$$

Proof. If χ_i is the characteristic function of the orbit U_i, then the functions $\xi_i = (1/|U_i|^{1/2})\chi_i$ for $i = 1, \ldots, h$, form an orthonormal basis for the space of K-invariant functions on X. Then from Proposition 1 it follows that

$$\eta(g) = \sum_{i=1}^{h} \sum_{x \in X} \xi_i(g^{-1}x)\xi_i(x) = \sum_{i=1}^{h} \frac{1}{|U_i|} |gU_i \cap U_i|. \qquad \blacksquare$$

Now we recall some elementary facts from the representation theory of the symmetric group S_N [3, pp. 135–136 and 148]. The trivial representation of S_N is denoted by $S^{(N)}$. The space of all complex valued functions defined on the set $\{1, \ldots, N\}$ is denoted by $M^{(N-1,1)}$; the space of all functions $\phi \in M^{(N-1,1)}$ such that $\sum_{i=1}^{N} \phi(i) = 0$ is invariant and irreducible under the action of S_N and is denoted by $S^{(N-1,1)}$. The decomposition of $M^{(N-1,1)}$ into irreducible representations of S_N is $M^{(N-1,1)} = S^{(N)} \oplus S^{(N-1,1)}$. The space of all complex valued functions defined on the set $\{\{(i, j\}$: $i, j \in \{1, \ldots, N\}$ and $i \neq j\}$ (this is the family of all subsets of $\{1, \ldots, N\}$ of size 2) is denoted by $M^{(N-2,2)}$. For $i \in \{1, \ldots, N\}$, define $\vartheta_i \in M^{(N-2,2)}$ as the characteristic function of the set $\{\{i, j\}$: $j \in (\{1, \ldots, N\} \setminus \{i\}\}$. Then the subspace of $M^{(N-2,2)}$ spanned by $\{\vartheta_1, \ldots, \vartheta_N\}$ is S_N-invariant, isomorphic to $M^{(N-1,1)}$, and its orthogonal complement is irreducible under the action of S_N and is denoted by $S^{(N-2,2)}$. So we have the following decomposition into irreducibles: $M^{(N-2,2)} = S^{(N)} \oplus S^{(N-1,N)} \oplus S^{(N-2,2)}$. Finally, we recall that $M^{(N-2,1,1)}$ denotes the space of all functions on the set $\{(i, j)$: $i, j \in \{1, \ldots, N\}$ and $i \neq j\}$ (this is the set of ordered pairs of distinct elements of $\{1, \ldots, N\}$). Moreover, there exists an irreducible representation of S_N denoted by $S^{(N-2,1,1)}$ such that $M^{(N-2,1,1)} = S^{(N)} \oplus 2S^{(N-1,1)} \oplus S^{(N-2,2)} \oplus S^{(N-2,1,1)}$. Now we want to compute the K-spherical characters of the preceding representations of S_N. First some notation: in what follows, A_h will denote the set $\{(h-1)n + 1, \ldots, hn\}$, i.e., the content of the urn h at the beginning of the process. Moreover, we define the functions R_{hk} on S_N by $R_{hk}(g) = |gA_h \cap A_k|$. We recall that K is the product $S_n \times \cdots \times S_n$ m times, where the hth S_n is the stabilizer of A_h.

THEOREM 3. *The K-spherical characters of the representations of S_N on $M^{(N-1,1)}$, $M^{(N-2,2)}$, and $M^{(N-2,1,1)}$ are given, respectively, by*

$$\eta_1 = \frac{1}{n}\sum_{h=1}^{m} R_{hh},$$

$$\eta_2 = \frac{1}{n(n-1)}\sum_{h=1}^{m} R_{hh}(R_{hh}-1) + \frac{1}{n^2}\sum_{1\le h<k\le m}(R_{hk}R_{kh} + R_{hh}R_{kk}),$$

$$\eta_{1,1} = \frac{1}{n(n-1)}\sum_{h=1}^{m} R_{hh}(R_{hh}-1) + \frac{2}{n^2}\sum_{1\le h<k\le m} R_{hh}R_{kk}.$$

Proof. We derive only the formula for η_2; the others can be obtained analogously. The orbits of K on the set $\{\{i,j\}: i,j \in \{1,\ldots,N\}: i \neq j\}$ are given by the subsets $U_h = \{\{i,j\}: i,j \in A_h\}$ and $U_{h,k} = \{\{i,j\}: i \in A_h$ and $j \in A_k\}$ with $h \neq k$. If $g \in S_N$, then $|gU_h \cap U_h| = (R_{hh}(g)(R_{hk}(g)-1))/2$ and $|gU_{h,k} \cap U_{h,k}| = R_{h,h}(g)R_{k,k}(g) + R_{h,k}(g)R_{k,h}(g)$. Then the formula for η_2 follows from Proposition 2. ∎

COROLLARY. *The K-spherical characters of the irreducible representations of S_N on the spaces $S^{(N-1,1)}$, $S^{(N-2,2)}$, and $S^{(N-2,1,1)}$ are given, respectively, by*

$$\phi_1 = \frac{1}{n}\sum_{h=1}^{m} R_{hh} - 1,$$

$$\phi_2 = \frac{1}{n(n-1)}\sum_{h=1}^{m} R_{hh}^2 - \frac{1}{n-1}\sum_{h=1}^{m} R_{hh} + \frac{1}{n^2}\sum_{1\le h<k\le m}(R_{hh}R_{kk} + R_{hk}R_{kh}),$$

$$\phi_{1,1} = \frac{1}{n^2}\sum_{1\le h<k\le m}(R_{hh}R_{kk} - R_{hk}R_{kh}) - \frac{1}{n}\sum_{h=1}^{m} R_{hh} + 1.$$

Proof. From the decomposition of the spaces $M^{(N-1,1)}$, $M^{(N-2,2)}$, and $M^{(N-2,1,1)}$ into irreducibles it follows that $\eta_1 = 1 + \phi_1$, $\eta_2 = \eta_1 + \phi_2$, and $\eta_{1,1} = \eta_2 + \phi_1 + \phi_{1,1}$. From these the formulas of the corollary follows easily. ∎

In order to prove the lower bound for the Bernoulli–Laplace diffusion model with many urns along the lines of [8], we need to compute the moment $\sum_{g \in G}[\phi_1(g)]^2\mu(g)$. Unfortunately, for $m \geq 3$, $(\phi_1)^2$ in general seems not to be a linear combination of 1, ϕ_1, ϕ_2, and $\phi_{1,1}$. Tedious but elementary calculations in the case $m = 3$ show this. So we need the following lemma. We recall that $N = nm$ and $n, m \geq 2$.

LEMMA 5. *Let $\psi_h \in M^{(N-2,2)}$ be the characteristic function of the set $\{\{i, j\}: i, j \in A_h, i \neq j\}$, $h = 1, \dots, m$. Let σ_h, ζ_h, and τ_h be the orthogonal projections of ψ_h, respectively, on $S^{(N-2,2)}$, $S^{(N-1,1)}$, and $S^{(N)}$. Then the norms of such projections as functions of $M^{(N-2,2)}$ are given by*

$$\|\sigma_h\|^2 = \frac{n(n-1)(N-n)(N-n-1)}{2(N-1)(N-2)},$$

$$\|\zeta_h\|^2 = \frac{n(n-1)^2(N-n)}{N(N-2)},$$

$$\|\tau_h\|^2 = \frac{[n(n-1)]^2}{2N(N-1)},$$

for $h = 1, \dots, m$.

Proof. First, we look for a decomposition $\psi_h = \sigma_h + \omega_h$ with $\sigma_h \in S^{(N-2,2)}$ and $\omega_h \in M^{(N-1,1)}$ of the form $\omega_h = \alpha \sum_{i \in A_h} \vartheta_i + \beta$, $\alpha, \beta \in \mathbb{C}$ to be determined (we recall that ϑ_i denotes the characteristic function of the set $\{\{i, j\}: j \in (\{1, \dots, N\} \setminus \{i\})\}$). However, $\psi_h - \omega_h$ belongs to the space $S^{(N-2,2)}$ iff $\langle \psi_h - \omega_h, \vartheta_j \rangle = 0$ for $j = 1, \dots, N$. If $j \in A_h$, then $\langle \psi_h - \omega_h, \vartheta_j \rangle = n - 1 - \alpha(n - 1 + N - 1) - \beta(N-1)$ and if $j \notin A_h$, then $\langle \psi_h - \omega_h, \vartheta_j \rangle = -\alpha n - \beta(N-1)$. From this we can find the values of α and β: $\alpha = (n-1)/(N-2)$ and $\beta = -(n(n-1))/((N-1)(N-2))$. Next we look for a decomposition $\omega_h = \zeta_h + \tau_h$ with $\zeta_h \in S^{(N-1,1)}$ and $\tau_h \in S^{(N)}$. Whereas $S^{(N)}$ is the space of constant functions in $M^{(N-2,2)}$, if we denote by $\mathbf{1}$ the constant function identically 1, then τ_h is the constant function identically equal to

$$\begin{aligned}
\frac{2}{N(N-1)}\langle \omega_h, \mathbf{1}\rangle
&= \frac{2}{N(N-1)}\left[\frac{n-1}{n-2}\sum_{i\in A_h}\langle \vartheta_i, \mathbf{1}\rangle - \frac{n(n-1)}{(N-2)(N-1)}\langle \mathbf{1}, \mathbf{1}\rangle\right]\\
&= \frac{n(n-1)}{N(N-1)}.
\end{aligned}$$

So

$$\zeta_h = \omega_h - \tau_h = \frac{n-1}{N-2}\sum_{i\in A_h}\vartheta_i - \frac{2n(n-1)}{N(N-2)}.$$

Moreover σ_h can be obtained as a difference $\sigma_h = \psi_h - \omega_h$. Now we compute the norms of the preceding projections of ψ_h. First, it is clear that $\|\psi_h\|^2 = (n(n-1))/2$ and $\|\tau_h\|^2 = [n(n-1)]^2/(2N(N-1))$. Moreover,

$$\begin{aligned}
\|\zeta_h\|^2 &= \left(\frac{n-1}{N-2}\right)^2\left[\sum_{i\in A_h}\langle \vartheta_i, \vartheta_i\rangle + 2\sum_{\substack{i,j\in A_h\\ i<j}}\langle \vartheta_i, \vartheta_j\rangle\right.\\
&\qquad\qquad\left. - \frac{4n}{N}\sum_{j\in A_h}\langle \vartheta_j, \mathbf{1}\rangle + \frac{4n^2}{N^2}\langle \mathbf{1}, \mathbf{1}\rangle\right]\\
&= \frac{n(n-1)^2(N-n)}{N(N-2)}.
\end{aligned}$$

Finally,

$$\|\sigma_h\|^2 = \|\psi_h\|^2 - \|\tau_h\|^2 - \|\zeta_h\|^2 = \frac{n(n-1)(N-n)(N-n-1)}{2(N-1)(N-2)}. \qquad \blacksquare$$

COROLLARY 1. *If* μ *is the probability measure of Sections* 2 *and* 3, *then, for* $h = 1, \dots, m$,

$$\sum_{g \in S_N} \mu^{*k}(g) R_{hh}(g)^2 = \frac{n^2(n-2+m)}{m(N-1)} + \frac{n^2(n-1)(m-1)[n(m-1)-1]}{(N-1)(N-2)} \times \left[1 - \frac{4(N-1)}{n^2 m(m-1)}\right]^k + \frac{n^2(m-1)(m+2n-4)}{m(N-2)} \times \left[1 - \frac{2}{n(m-1)}\right]^k.$$

Proof. The matrix coefficient $\langle g\psi_h, \psi_h \rangle = \sum_{x \in X} \psi_h(g^{-1}x)\psi_h(x)$, where X is the set $\{\{i,j\}: i, j \in \{1, \dots, N\}, i \neq j\}$, is equal to $(R_{hh}(g)(R_{hh}(g) - 1))/2$. However, $\langle g\psi_h, \psi_h \rangle = \langle g\sigma_h, \sigma_h \rangle + \langle g\zeta_h, \zeta_h \rangle + \langle g\tau_h, \tau_h \rangle$, where σ_h, ζ_h, and τ_h are as in the lemma. Moreover, from (2.1) it follows that

$$\sum_{g \in S_N} \frac{1}{\|\sigma_h\|^2} \langle g\sigma_h, \sigma_h \rangle \mu^{*k}(g) = q((N-2,2))^k = \left[1 - \frac{4(N-1)}{n^2 m(m-1)}\right]^k,$$

$$\sum_{g \in S_N} \frac{1}{\|\zeta_h\|^2} \langle g\zeta_h, \zeta_h \rangle \mu^{*k}(g) = q((N-1,1))^k = \left[1 - \frac{2}{n(m-1)}\right]^k,$$

$$\sum_{g \in S_N} \frac{1}{\|\tau_h\|^2} \langle g\tau_h, \tau_h \rangle \mu^{*k}(g) = 1.$$

Finally, again from (2.1) and from the formula for ϕ_1 in the corollary to Theorem 3, it follows that

$$\sum_{g \in S_N} R_{hh}(g) \mu^{*k}(g) = \frac{n}{m} + \frac{m-1}{m} n \left[1 - \frac{2}{n(m-1)}\right]^k.$$

Collecting together all these formulas and Lemma 5, the corollary follows. ∎

COROLLARY 2. *If we define* $f = ((N-1)/(m-1))^{1/2}\phi_1$, *then*

$$\sum_{g\in S_N} f(g)^2 \mu^{*k}(g) = \left\{(N-1)\frac{m}{2} - \frac{m[n(m-1)-1]}{N-2}\right\} \times\left[1-\frac{4(N-1)}{n^2m(m-1)}\right]^k + \frac{(m-2)(N-1)}{2}\left[1-\frac{4}{n(m-1)}\right]^k + \frac{(N-1)(m-2)}{N-2}\left[1-\frac{2}{n(m-1)}\right]^k + 1.$$

Proof. From the corollary to Theorem 3 it is easy to deduce that

$$(\phi_1)^2 = \phi_2 + \phi_{1,1} - \frac{1}{n^2(n-1)}\sum_{h=1}^{m}(R_{hh})^2 + \frac{1}{n(n-1)}\sum_{h=1}^{m} R_{hh}.$$

Then using the previous corollary, the formula follows. ∎

5. THE LOWER BOUND

Following [8], we prove a lower bound using the normalized spherical character f of the last corollary as a random variable. [There is a small mistake in [8]. The normalized spherical function should be $(2n-1)^{1/2}\phi_1$.]

THEOREM 4. *Let* P_k *and* U *be as in Theorems* 1 *and* 2. *Then there exists a universal positive constant* b_3 *such that*

$$\textit{if } k = \tfrac{1}{4}n(m-1)\big(\log(nm^2) - c\big) \textit{ with } c \in \left[0, \tfrac{1}{2}\log(nm)\right], \textit{ then}$$

$$\|P_k - U\| \geq 1 - b_3 e^{-c/2}.$$

Proof. The first moment $E_k(f)$ of f under μ^{*k} can be easily computed using (2.1):

$$E_k(f) = \sum_{g \in S_N} f(g)\mu^{*k}(g) = [(m-1)(N-1)]^{1/2}\left[1 - \frac{2}{n(m-1)}\right]^k.$$

The variance $\mathrm{Var}_k(f)$ of f under μ^{*k} can be computed using the last corollary of the previous section:

$$\begin{aligned}\mathrm{Var}_k(f) = {} & \left\{(N-1)\frac{m}{2} - \frac{m[n(m-1)-1]}{N-2}\right\}\left[1 - \frac{4(N-1)}{n^2m(m-1)}\right]^k \\ & + \frac{(m-2)(N-1)}{2}\left[1 - \frac{4}{n(m-1)}\right]^k \\ & + \frac{(N-1)(m-2)}{N-2}\left[1 - \frac{2}{n(m-1)}\right]^k \\ & + 1 - (m-1)(N-1)\left[1 - \frac{2}{n(m-1)}\right]^{2k}.\end{aligned}$$

When $k = \frac{1}{4}n(m-1)(\log(nm^2) - c)$ with $c > 0$, these expressions become

$$E_k(f) = \exp\left[\frac{c}{2} + O\left(\frac{\log(nm^2)}{nm}\right) + O\left(\frac{c}{nm}\right)\right]$$

and

$$\begin{aligned}\mathrm{Var}_k(f) = {} & 1 + e^c\left[O\left(\frac{c}{nm}\right) + O\left(\frac{\log(nm^2)}{nm}\right)\right] \\ & + \frac{(nm-1)(m-2)e^{c/2}}{(nm-2)m(n)^{1/2}}\exp\left[O\left(\frac{\log(nm^2)}{nm}\right) + O\left(\frac{c}{nm}\right)\right] \\ & - \frac{[n(m-1)-1]e^c}{(nm-2)nm}\exp\left[O\left(\frac{\log(nm^2)}{nm}\right) + O\left(\frac{c}{nm}\right)\right].\end{aligned}$$

Thus there exist two universal constants H_1 and H_2 such that for $c \in [0, \frac{1}{2}\log(nm)]$ and $k = \frac{1}{4}n(m-1)(\log(nm^2) - c)$, $\mathrm{Var}_k(f) \le H_1 + H_2 e^{c/2}$. If we define, for $\alpha \in [0, \frac{1}{2}e^{c/2}]$, $A_\alpha = \{g \in S_N\colon\ |f(g)| \le \alpha\}$, then by

Chebyshev's inequality we obtain

$$\mu^{*k}(A_\alpha) \leq \mathrm{Var}_k(f)/(E_k(f) - \alpha)^2 \leq H_3(H_1 + H_2 e^{c/2})/(e^{c/2} - \alpha)^2,$$

where H_3 is uniformly bounded for $0 \leq c \leq \frac{1}{2}\log(nm)$. Moreover, from the orthogonality relations [3, p. 11] it follows that under the uniform probability measure V on S_N, $\mathrm{Var}_V(f) = 1$. Thus from Chebyshev's inequality, $V(A_\alpha) \geq 1 - 1/\alpha^2$ and so

$$|V(A_\alpha) - \mu^{*k}(A_\alpha)| \geq 1 - \frac{1}{\alpha^2} - \frac{H_1 + H_2 e^{c/2}}{(e^{c/2} - \alpha)^2} H_3.$$

The theorem follows choosing $\alpha = \frac{1}{2}e^{c/2}$. (Note that V and μ^{*k} are the bi-K-invariant functions on S_N that correspond to U and P_k and that A_α is K-invariant.) ∎

Remark 3. In [5] and [6] Diaconis and Saloff-Coste developed comparison techniques that give bounds on the eigenvalues of a reversible Markov chain (in [5] this is a random walk on a finite group) in terms of the eigenvalues of a second chain. In many examples, the comparison chains are those studied in [7] and [8]. Because we give a generalization of the two last cited papers, it would be interesting to obtain results for more complicated chains using the present paper and the techniques of Diaconis and Saloff-Coste.

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